

Homotopical Khovanov homology

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Abstract

We modify the definition of the Khovanov complex for oriented links in a thickening of an oriented surface to obtain a triply graded homological link invariant with a new homotopical grading.

1 Introduction

Virtual knot theory was discovered by Kauffman in [5]. It turned out that virtual links are isotopy classes of links in thickened surfaces considered up to stabilizations.

Thus, virtual knots and links possess properties coming from knots as well as those coming from curves on 2-surfaces.

It turned out [8, 2, 6] that this homotopical information can be converted into coefficients of polynomials, or, in other words, polynomials can be thought of to have homotopy classes as coefficients.

Homotopy classes of curves in a closed 2-surface can be thought of as conjugacy classes of the fundamental group of the surface. The recognition problem for such curves and groups is solved, see, e.g. [4].

Some numerical information coming from these homotopy classes can be converted into additional gradings of the Khovanov homology [3, 7].

In the present paper, we show how the homotopical information about curves in Kauffman states can be converted into additional gradings.

Thus, curves which appear as summands (with some other coefficients) are raised to the other level. Though some factorizations in the grading space are needed for the complex to be well defined, the homotopy information which is converted into gradings, does not reduce to some homology classes or numerical grading; highly non-commutative group theory comes into play in the new form of gradings of the Khovanov complex.

2 Homotopical Khovanov homology

Let S be a connected closed oriented two dimensional surface. We consider links in the thickening $S \times [0, 1]$ of the surface. Such links can be described by diagrams look like 4-valent graphs embedded into S with the structure

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of over- and undercrossing in the vertices of the graphs. Different diagrams of the same link differ by diagram isotopies and Reidemeister moves.

Definition 1. Let D be a link diagram in S . A *source-sink structure* on D is an orientation of edges of D (considered as a 4-valent graph) such that at each crossing of D there are two opposite incoming edges and two opposite outgoing edges (see Fig. 1).

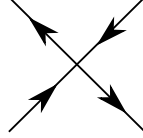


Figure 1: Source-sink structure

Many invariants of classical knots and links can be extended to the case of links in the surface S with minor changes of construction or without changes at all. Let us recall the construction of Khovanov homology of links. For simplicity we work over \mathbb{Z}_2 .

Let D be a diagram of some oriented link L in the surface S . Let $\mathcal{X}(D)$ be the set of crossing of the diagram D . At each crossing of D there are two possible resolutions 0 and 1 (see Fig. 2). A *state* is a map $s: \mathcal{X}(D) \rightarrow \{0, 1\}$ (or equivalently a sequence of 0s and 1s indexed by crossings of D). Any state s determines resolutions at all crossings of the diagram D . The diagram D_s which appears after resolution at each crossing according to the state s , has no crossings, and therefore it is a set of nonintersecting circles (closed simple curves) in S . Let $\gamma(s)$ be the number of circles in D_s and $\beta(s)$ be the number of 1s in s .



Figure 2: Resolutions of a crossing

Thus, D has 2^n states, where n is the number of crossings in D . The set of states $\{0, 1\}^{\mathcal{X}(D)}$ can be interpreted as the set of vertices of a n -dimensional cube (*the state cube*). The edges of the cube are oriented from the state $(0, 0, \dots, 0)$ to the state $(1, 1, \dots, 1)$.

Let V be the two dimensional graded vector space with basis v_- and v_+ . Set the grading of the basis vectors to be $\deg(v_{\pm}) = \pm 1$. Let the maps $m: V \otimes V \rightarrow V$ and $\Delta: V \rightarrow V \otimes V$ be given by formulas

$$\begin{aligned} m(v_+ \otimes v_+) &= v_+, & m(v_+ \otimes v_-) &= v_-, & \Delta(v_-) &= v_- \otimes v_- \\ m(v_- \otimes v_+) &= v_-, & m(v_- \otimes v_-) &= 0, & \Delta(v_+) &= v_+ \otimes v_- - v_- \otimes v_+ \end{aligned} \quad (1)$$

With each state s , assign the state a space $V(s) = V^{\otimes \gamma(s)}$. Now, the chain space of the Khovanov complex, we are going to construct for D , will be the sum

$$[[D]] = \bigoplus_{s \in \{0,1\}^{\mathcal{X}(D)}} V(s).$$

The space $[[D]]$ has two gradings: the *homological grading* β and the *quantum grading* q . If $x = x_1 \otimes x_2 \otimes \cdots \otimes x_{\gamma(s)} \in V(s) \subset [[D]]$ where $x_i = v_{\pm}, i = 1, \dots, \gamma(s)$, then we set $\beta(x) = \beta(s)$ and $q(x) = \sum_{i=1}^{\gamma(s)} \deg(x_i) + \beta(s)$.

Any edge $e = s \rightarrow s'$ connects two states s and s' . The resolutions D_s and $D_{s'}$ differ at one crossing. If D admits a source-sink structure then one of the following pictures can happen: either a circle can split into two circles or two circles can merge into one circle (see Fig. 3).

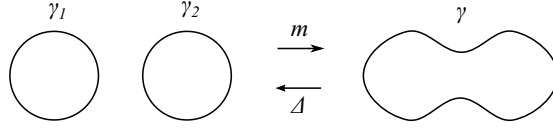


Figure 3: Transformations of resolution

In the first case, one should consider the map $\Delta \otimes \text{id}^{\otimes \gamma(s)-1} : V(s) \rightarrow V(s')$ where Δ acts on the factor V which corresponds to the splitting circle. In the second case, one should consider the map $m \otimes \text{id}^{\otimes \gamma(s)-2} : V(s) \rightarrow V(s')$ where m acts on the factors $V \otimes V$ which correspond to the merging circles. The considered map from $V(s)$ to $V(s')$ is called the *partial differential* corresponding to the edge e of the Khovanov complex and denoted as ∂_e .

The sum of partial differentials for all the edges of the cube of states $d = \sum_{e=s \rightarrow s'} \partial_e$ is the differential of Khovanov complex. The differential does not change quantum grading and increases homology grading by 1.

In order to construct an invariant homology theory, we should ensure among other thing invariance under first Reidemeister moves. To this end, let us shift the homological grading by $-n_-$ and the quantum grading by $n_+ - 2n_-$ where n_+ and n_- are the numbers of positive and negative crossings in the diagram D . Let $C(D)$ be the shifted complex. Then homology $Kh(D) = H(C(D), d)$ is the *Khovanov homology* of the link L .

Definition 2. Let S be a closed two-dimensional surface. We consider the set $\mathfrak{L} = [S^1; S]$ of all the homotopy classes of free oriented loops in S . Let $\bigcirc \in \mathfrak{L}$ be the homotopy class of contractible loops.

For any closed curve γ one can consider the curve $-\gamma$ obtained from γ by the orientation change. Let \mathfrak{H} be the quotient group of the free abelian group with generator set \mathfrak{L} modulo the relations $\bigcirc = 0$ and $[\gamma] = [-\gamma]$ for all free loops γ .

We consider a new homotopical grading in the complex $[[L]]$ valued in the group \mathfrak{H} . Let s be a state and $C_1, C_2, \dots, C_{\gamma(s)}$ be the circles of the resolution L_s . For each element $x = x_1 \otimes x_2 \otimes \cdots \otimes x_{\gamma(s)} \in V(s)$

where $x_i = v_{\pm}$ corresponds to the circle $C_i, i = 1, \dots, \gamma(s)$, we define its *homotopical grading* as

$$h(x) = \sum_{i=1}^{\gamma(s)} \deg(x_i)[C_i] \in \mathfrak{H}.$$

Let d_h be the "part" of the differential d that does not change the homotopical grading (i.e. the composition of d with the projection to the subspace with the correspondent homotopical grading). The map d_h is presented as a sum $d_h = \sum_{e \in E} (\partial_e)_h$ over all the edges of the state cube. For any edge $e = s \rightarrow s'$ the partial map $(\partial_e)_h$ has the form $m_h \otimes \text{id}^{\otimes \gamma(s)-2}$ if the partial differential is equal to $m \otimes \text{id}^{\otimes \gamma(s)-2}$ and $(\partial_e)_h$ has the form $\Delta_h \otimes \text{id}^{\otimes \gamma(s)-1}$ if the partial differential is equal to $\Delta \otimes \text{id}^{\otimes \gamma(s)-1}$. Here m_h and Δ_h are the parts of the maps m and Δ that preserve the homotopical grading.

The formula for the map $m_h: V \otimes V \rightarrow V$, which corresponds to the bifurcation of two circles γ_1 and γ_2 to a circle γ , depends on the homotopy classes of the circles:

$$m_h = \begin{cases} m, & [\gamma_1] = [\gamma_2] = [\gamma] = \bigcirc; \\ m_h^1, & [\gamma_2] = \bigcirc, [\gamma_1] = [\gamma] \neq \bigcirc; \\ m_h^2, & [\gamma_1] = \bigcirc, [\gamma_2] = [\gamma] \neq \bigcirc; \\ m_h^0, & [\gamma] = \bigcirc, [\gamma_1], [\gamma_2] \neq \bigcirc; \\ 0, & [\gamma_1], [\gamma_2], [\gamma] \neq \bigcirc. \end{cases} \quad (2)$$

The maps m_h^0, m_h^1, m_h^2 are defined by the formulas

$$\begin{aligned} m_h^0(v_+ \otimes v_+) &= 0, & m_h^1(v_+ \otimes v_+) &= v_+, & m_h^2(v_+ \otimes v_+) &= v_+, \\ m_h^0(v_+ \otimes v_-) &= v_-, & m_h^1(v_+ \otimes v_-) &= 0, & m_h^2(v_+ \otimes v_-) &= v_-, \\ m_h^0(v_- \otimes v_+) &= v_-, & m_h^1(v_- \otimes v_+) &= v_-, & m_h^2(v_- \otimes v_+) &= 0, \\ m_h^0(v_- \otimes v_-) &= 0, & m_h^1(v_- \otimes v_-) &= 0, & m_h^2(v_- \otimes v_-) &= 0. \end{aligned} \quad (3)$$

Analogously, the formula for the map $\Delta_h: V \rightarrow V \otimes V$, which corresponds to transformations of a circle γ to two circles γ_1 and γ_2 , depends on the homotopical classes of the circles as follows:

$$\Delta_v = \begin{cases} \Delta, & [\gamma_1] = [\gamma_2] = [\gamma] = \bigcirc; \\ \Delta_h^1, & [\gamma_2] = \bigcirc, [\gamma_1] = [\gamma] \neq \bigcirc; \\ \Delta_h^2, & [\gamma_1] = \bigcirc, [\gamma_2] = [\gamma] \neq \bigcirc; \\ \Delta_h^0, & [\gamma] = \bigcirc, [\gamma_1], [\gamma_2] \neq \bigcirc; \\ 0, & [\gamma_1], [\gamma_2], [\gamma] \neq \bigcirc. \end{cases} \quad (4)$$

The maps $\Delta_h^0, \Delta_h^1, \Delta_h^2$ are defined by the formulas

$$\begin{aligned} \Delta_h^0(v_+) &= v_+ \otimes v_- + v_- \otimes v_+, & \Delta_h^1(v_+) &= v_+ \otimes v_-, & \Delta_h^2(v_+) &= v_- \otimes v_+, \\ \Delta_h^0(v_-) &= 0, & \Delta_h^1(v_-) &= v_- \otimes v_-, & \Delta_h^2(v_-) &= v_- \otimes v_-. \end{aligned} \quad (5)$$

Remark 1. The formulas (2),(4) show that a partial map is not zero only if one of the transformed circles is not trivial.

Theorem 1. *The map d_h is a differential.*

Proof. In order to show that the map d_h is a differential we need to check that every 2-face in the state cube of the diagram D is commutative. Any 2-face corresponds to a resolution switch of two crossings of the diagram whereas the resolution of the other crossings is fixed. Since any resolution of all the crossings of the diagram except the chosen two crossings yields a diagram D' with two crossings and source-sink structure on it, there three possible configurations of the two remaining crossings' position (see Fig. 4). Note that any (partial) resolution of a diagram with source-sink structure inherits source-sink structure from the diagram. Here we consider only the components of the diagram D' which contain the crossings and don't point out under- and overcrossings structure because it doesn't change the states of the 2-face (but determines the maps of the face).

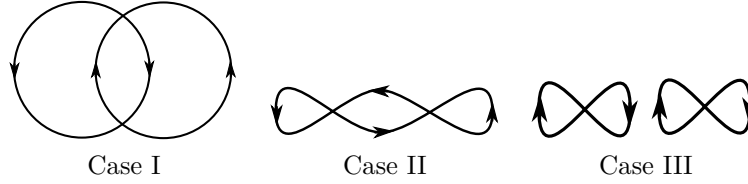


Figure 4: Diagrams with two crossings

If the crossings belong to different components (case III) then the 2-face is obviously commutative.

Case I. Among the resolutions of the face there are two states with two circles and two states with one circle (see Fig. 5). Thus, we have two subcases depending on how many circles are there in the initial state: with one circle (for example, e) or with two circles (a and b). The corresponding diagrams are shown in Fig. 6. Now we should check all the variants, which homotopy classes of the circles a, b, c, d, e, f are trivial and which are not.

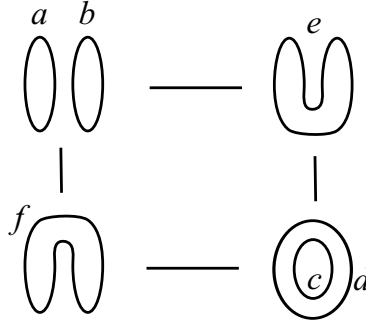


Figure 5: Case I: Resolutions

Case I.A. The initial state of the face has one circle.

Let k be the number of homotopically trivial circles in the pair a, b , and l be the number of homotopically trivial circles in the pair c, d . Then $0 \leq k, l \leq 2$. Without loss of generality assume that $k \leq l$.

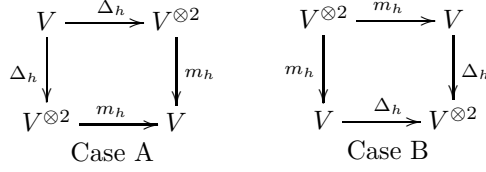


Figure 6: Case I: Face diagrams

If $k = l$ then the 2-face is commutative due to symmetry.

Let $k = 0, l = 1$. Assume that the circle c is trivial and d is not trivial. Then $[e] = [f] = [d]$ so e and f are not trivial. Hence, the commutativity relation reduces to the equality $m_h^2 \Delta_h^2 = 0$. But $m_h^2 \Delta_h^2(v_\pm) = m_h^2(v_- \otimes v_\pm) = 0$, so the face is commutative.

Let $k = 0, l = 2$. Then e and f are trivial as connected sums of trivial circles c and d . Hence, we need check the equality $m_h^0 \Delta_h^0 = m \Delta$. But the both maps are zero (over \mathbb{Z}_2), so the face is commutative.

The case $k = 1, l = 2$ is impossible. Indeed, let $[a] = \bigcirc$ and $[b] \neq \bigcirc$. Then $[e] = [b]$ is not trivial. On the other hand e must be trivial as a connected sum of trivial circles c and d .

Case I.B. The initial state of the face has two circles.

If e and f are either both trivial or both nontrivial then the face is commutative due to symmetry.

Let $[e] = \bigcirc$ and $[f] \neq \bigcirc$. Then the circles a, b, c, d must be all nontrivial. Commutativity of the face follows from the equality $\Delta_h^0 m_h^0 = 0$.

Case II. Among the resolutions of the face there are one state with one circle, two states with two circles and two states with one circle and one state with three circles (see Fig. 7). There are three subcases (A,B,C) depending on how many circles are there in the initial state. The corresponding diagrams are shown in Fig. 8.

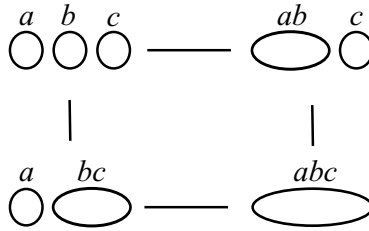


Figure 7: Case II: resolutions

Let us enumerate possible variants when some of the circles a, b, c, ab, bc, abc are homotopically trivial:

1. The circles a, b, c are trivial.

Then the circles ab, bc, abc are trivial and we have a face of the usual Khovanov complex.

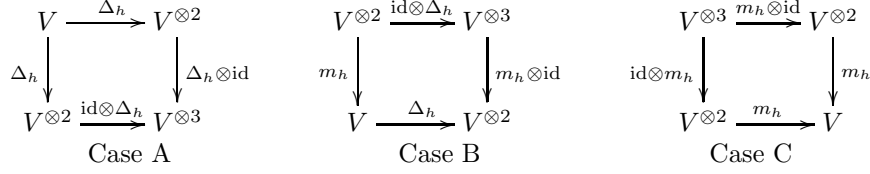


Figure 8: Case II: face diagrams

2. Among a, b, c there is only one nontrivial circle.

- (a) $[a] \neq \bigcirc$
- (b) $[b] \neq \bigcirc$
- (c) $[c] \neq \bigcirc$

In this case $[abc] \neq \bigcirc$, the circle ab is not trivial if $[a] \neq \bigcirc$ or $[b] \neq \bigcirc$, and the circle bc is not trivial if $[b] \neq \bigcirc$ or $[c] \neq \bigcirc$.

3. Among a, b, c only one circle is trivial.

- (a) $[a] = \bigcirc$. Then $[ab] = [b]$ is not trivial and $[abc] = [bc]$.
 - i. $[bc] = \bigcirc$
 - ii. $[bc] \neq \bigcirc$
- (b) $[b] = \bigcirc$. Then $[ab] = [a]$, $[bc] = [c]$ are not trivial.
 - i. $[abc] = \bigcirc$
 - ii. $[abc] \neq \bigcirc$
- (c) $[c] = \bigcirc$. Then $[bc] = [b]$ is not trivial and $[abc] = [ab]$.
 - i. $[ab] = \bigcirc$
 - ii. $[ab] \neq \bigcirc$

4. None of a, b, c is trivial.

- (a) $[ab] = [bc] = \bigcirc$. Then $[abc] = [a] = [c]$ is not trivial.
- (b) $[ab] = \bigcirc, [bc] \neq \bigcirc$. Then $[abc] = [c]$ is not trivial.
- (c) $[ab] \neq \bigcirc, [bc] = \bigcirc$. Then $[abc] = [a]$ is not trivial.
- (d) $[ab], [bc] \neq \bigcirc$.
 - i. $[abc] = \bigcirc$
 - ii. $[abc] \neq \bigcirc$

Any combination of cases A,B,C and 1–4.d.ii determines the maps of the face. Thus, we can write down commutativity relation of the face in table form (see Table 1). It is easy to check that all the equalities of the table hold, so the case II is also commutative. The proof is finished. \square

Remark 2. 1. The differentials d and d_h don't commute. Indeed, if this were a case then the difference $d_v = d - d_h$ would have been a differential. But in the case I.B considered above in Theorem 1 with trivial e and nontrivial f we have $v_+ \otimes v_- \mapsto 0 \mapsto 0$ for the path of the face that goes through e , and $v_+ \otimes v_- \mapsto v_- \mapsto v_- \otimes v_-$ for the path of the face that goes through f . So, the face is not commutative for the map d_v .

Table 1. Commutativity relation.

	A	B	C
1		as in usual Khovanov complex	
2.a	$(\Delta_h^1 \otimes \text{id})\Delta_h^1 = (\text{id} \otimes \Delta_h^1)\Delta_h^1$	$(m_h^1 \otimes \text{id})(\text{id} \otimes \Delta_h^1) = \Delta_h^1 m_h^1$	$m_h^1(m_h^1 \otimes \text{id}) = m_h^1(\text{id} \otimes m_h^1)$
2.b	$(\Delta_h^2 \otimes \text{id})\Delta_h^2 = (\text{id} \otimes \Delta_h^2)\Delta_h^2$	$(m_h^2 \otimes \text{id})(\text{id} \otimes \Delta_h^2) = \Delta_h^2 m_h^2$	$m_h^2(m_h^2 \otimes \text{id}) = m_h^2(\text{id} \otimes m_h^2)$
2.c	$(\Delta_h^0 \otimes \text{id})\Delta_h^2 = (\text{id} \otimes \Delta_h^2)\Delta_h^2$	$(m_h^0 \otimes \text{id})(\text{id} \otimes \Delta_h^2) = \Delta_h^2 m_h^0$	$m_h^2(m_h^0 \otimes \text{id}) = m_h^2(\text{id} \otimes m_h^0)$
3.a.i	$(\Delta_h^2 \otimes \text{id})\Delta_h^0 = (\text{id} \otimes \Delta_h^0)\Delta_h^2$	$(m_h^2 \otimes \text{id})(\text{id} \otimes \Delta_h^0) = \Delta_h^0 m_h^2$	$m_h^0(m_h^2 \otimes \text{id}) = m_h^0(\text{id} \otimes m_h^2)$
3.a.ii	$0 = 0$	$0 = 0$	$0 = 0$
3.b.i	$(\Delta_h^1 \otimes \text{id})\Delta_h^0 = (\text{id} \otimes \Delta_h^2)\Delta_h^0$	$(m_h^1 \otimes \text{id})(\text{id} \otimes \Delta_h^2) = \Delta_h^0 m_h^1$	$m_h^0(m_h^1 \otimes \text{id}) = m_h^0(\text{id} \otimes m_h^2)$
3.b.ii	$0 = 0$	$(m_h^1 \otimes \text{id})(\text{id} \otimes \Delta_h^2) = 0$	$0 = 0$
3.c.i	$(\Delta_h^0 \otimes \text{id})\Delta_h^1 = (\text{id} \otimes \Delta_h^1)\Delta_h^0$	$(m_h^0 \otimes \text{id})(\text{id} \otimes \Delta_h^1) = \Delta_h^0 m_h^0$	$m_h^0(m_h^0 \otimes \text{id}) = m_h^0(\text{id} \otimes m_h^1)$
3.c.ii	$0 = 0$	$0 = 0$	$0 = 0$
4.a	$(\Delta_h^0 \otimes \text{id})\Delta_h^2 = (\text{id} \otimes \Delta_h^0)\Delta_h^2$	$(m_h^0 \otimes \text{id})(\text{id} \otimes \Delta_h^0) = \Delta_h^2 m_h^1$	$m_h^2(m_h^0 \otimes \text{id}) = m_h^1(\text{id} \otimes m_h^0)$
4.b	$(\Delta_h^0 \otimes \text{id})\Delta_h^2 = 0$	$0 = 0$	$m_h^2(m_h^0 \otimes \text{id}) = 0$
4.c	$0 = (\text{id} \otimes \Delta_h^0)\Delta_h^2$	$0 = 0$	$0 = m_h^1(\text{id} \otimes m_h^0)$
4.d.i	$0 = 0$	$0 = \Delta_h^0 m_h^0$	$0 = 0$
4.d.ii		$0 = 0$	

2. If we set $m_h^0 = m_h^1 = m_h^2 = 0$ and $\Delta_h^0 = \Delta_h^1 = \Delta_h^2 = 0$ in formulas (2), (4) the modified map d_h will also be a differential but the complex will not be invariant under Reidemeister moves.

Thus, $([[D]], d_h)$ is a chain complex with triple gradings. We use the conventional shift by $-n_-$ of homological grading and the shift by $n_+ - 2n_-$ of quantum grading (where n_+ and n_- are the numbers of positive and negative crossings in the diagram) to obtain a chain complex $C(D)$ whose homology $Kh_h(D) = H(C(D), d_h)$ we call *homotopical Khovanov homology*. It is an abelian group with three gradings

$$Kh_h(D) = \sum_{i \in \mathbb{Z}, j \in \mathbb{Z}, \mathfrak{h} \in \mathfrak{H}} Kh_h(D)_{i,j,\mathfrak{h}}$$

(here i is the homology grading, j is the quantum grading and \mathfrak{h} is the homotopical grading) which turns out to be a link invariant like ordinary Khovanov homology.

Theorem 2. *Homology Kh_h is an invariant of oriented links which admits source-sink structure.*

Proof. Let D be a diagram of an oriented link with source-sink structure and let D' be the diagram that obtained from D by a first Reidemeister move. Then the complex $[[D']]$ looks like $[[D]] \xrightarrow{\Delta_h} [[D \sqcup \bigcirc]]$ (if the new crossing is negative) or $[[D \sqcup \bigcirc]] \xrightarrow{m_h} [[D]]$ (if the new crossing is positive). The complex $[[D \sqcup \bigcirc]]$ is isomorphic to $[[D]] \otimes V$. It contains a subcomplex $[[D \sqcup \bigcirc]]_{v_+} = [[D]] \otimes v_+$ and we denote $[[D \sqcup \bigcirc]]_{v_+=0} = [[D \sqcup \bigcirc]] / [[D]]_{v_+}$ the quotient complex by this subcomplex.

Lemma 1. *The maps $[[D \sqcup \bigcirc]]_{v_+} \xrightarrow{m_h} [[D]]$ and $[[D]] \xrightarrow{\Delta_h} [[D \sqcup \bigcirc]]_{v_+=0}$ are isomorphisms of complexes.*

Proof. Since the small circle is trivial, $m_h = m$ or $m_h = m_h^1$. In both cases m_h establishes an isomorphism between $V \otimes v_+$ and V that extends to the desired isomorphism of complexes. For the second map we have $\Delta_h = \Delta$ or Δ_h^1 which define an isomorphism between V and $V \otimes V/(v_+ = 0)$. \square

With the above lemma we can use the proof of invariance of the usual Khovanov homology (see, for example, [1]) without any changes. \square

Remark 3. In fact, homotopical Khovanov homology can be defined for links without source-sink structure. Indeed, in general case an additional (nonorientable) bifurcation type appears where one circle transforms to another circle (see Fig. 9).

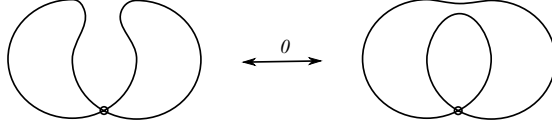


Figure 9: Nonorientable bifurcation of a circle

The circled crossings in Fig. 9,10 are considered as virtual but not real crossings, i.e. as additional intersections which appear by projection of the resolution into the plane.

In the state cube this bifurcation corresponds to zero map. So, in order to show that homotopical Khovanov complex is well defined we must check 2-faces which include the new type of state transformation (see Fig. 10).

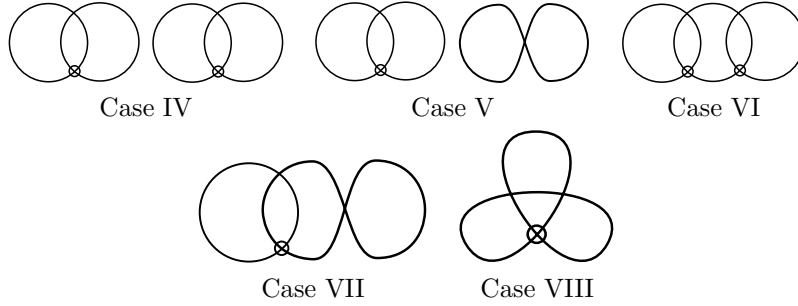


Figure 10: Nonorientable diagrams with two crossings

The commutativity relation for these faces is reduced to the equality $0 = 0$ for all cases except one (see Fig. 11) where it takes the form $m_h \Delta_h = 0$. But this equality holds because $m \Delta = m_h^0 \Delta_h^0 = m_h^1 \Delta_h^1 = m_h^2 \Delta_h^2 = 0$.

$$\begin{array}{ccc} V & \xrightarrow{\Delta_h} & V^{\otimes 2} \\ 0 \downarrow & & \downarrow m_h \\ V & \xrightarrow{0} & V \end{array}$$

Figure 11: Case VIII: nontrivial face diagram

The proof of invariance of homotopical Khovanov homology needs no corrections since it does not encounter the new bifurcation.

Thus, the following theorem holds.

Theorem 3. *Homology Kh_h is an invariant of oriented links.*

Let us mention some simple properties of homotopical Khovanov homology.

Proposition 1. *If L is a classical link in S , i.e. there is a 3-ball $B \subset S \times [0, 1]$ which contains L , then homotopical Khovanov homology of L coincides with the usual Khovanov homology of L .*

Proof. If L is a classical link then the homotopical grading of all elements in the Khovanov complex is zero, so maps m_h and Δ_h coincide with m and Δ . Hence, in this case we have the usual Khovanov complex. \square

Proposition 2. *Let L be an oriented link in S . Let $-L$ differ from L by orientation change and \bar{L} be the link with the opposite source-sink structure. Then $Kh_h(-L)_{i,j,\mathfrak{h}} = Kh_h(L)_{i,j,\mathfrak{h}}$, $Kh_h(\bar{L})_{i,j,\mathfrak{h}} = Kh_h(L)_{-i,-j,-\mathfrak{h}}$ and $Kh_h(\check{L})_{i,j,\mathfrak{h}} = Kh_h(L)_{i,j,\mathfrak{h}}$ for all $i \in \mathbb{Z}, j \in \mathbb{Z}, \mathfrak{h} \in \mathfrak{H}$. In other words, homotopical Khovanov homology does not depend on the choice of source-sink structure and does not detect invertibility of knots.*

The proof of the statement does not differ from the the proof for the usual Khovanov homology.

Example 1 (Simple closed curves in the surface). Let γ be a simple closed curve in S . It can be considered as a knot diagram without crossings. Usual Khovanov homology of γ is $Kh(\gamma) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with homological grading equal to 0 and quantum gradings equal to ± 1 . This homology does not depend on how the curve lie in the surface. Homotopical Khovanov homology is more sensitive. $Kh_h(\gamma) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ where the first summand has homological grading 0, quantum grading -1 and homotopical grading $-\lceil \gamma \rceil$, and the second summand has homological grading 0, quantum grading 1 and homotopical grading $\lceil \gamma \rceil$. Thus, the homotopy type of γ is kept in the homotopical grading.

Theorem 4. *Let knots K and K' in S have complexity 0, i.e. admit diagrams without crossings. Then K isotopic to K' (up to orientation reversion) if and only if $Kh_h(K) = Kh_h(K')$.*

Proof. Let K and K' are presented by simple closed curve in S . If $Kh_h(K) = Kh_h(K')$ then $[K] = [K'] \in \mathfrak{H}$ and $[K] = [\pm K'] \in \mathfrak{L}$. Hence there is a homotopy between simple closed curves K and K' . Then by Baer–Zieschang theorem there is an isotopy of S that sends K to K' . \square

Example 2 (Knots in torus). Let K be a knot in the thickening of torus T^2 . For torus we have $\mathfrak{L} = H_1(T^2, \mathbb{Z})$. Let D be a diagram of K in the torus.

For any state s the resolution D_s is a set of nonintersecting circles in T^2 . If γ_1 and γ_2 are nontrivial circles in D_s then their homology classes coincide up to sign. Then homotopy gradings of elements in $V(s)$ equal to $k[\gamma_1], k \in \mathbb{Z}$. If the homology class $[D] = [K]$ of the source-sink structure of the knot is not zero the gradings will be proportional to $[K]$. If the homology class $[D]$ is trivial then gradings can be nonproportional but must be even.

Consider a link L in the torus with two crossings (see Fig. 12). Homotopical Khovanov homology of the link L are isomorphic to $(\mathbb{Z}_2)^8$. The

gradings of generators of $Kh_h(L)$ are $(-1, -2, 0)$, $(0, 0, 0)$, $(0, 0, 0)$, $(1, 2, 0)$ and $(0, 2, 2[\lambda])$, $(0, -2, -2[\lambda])$, $(0, 2, 2[\mu])$, $(0, -2, -2[\mu])$ where λ and μ are the longitude and the meridian of the torus. Remark that the link L as a virtual link is a virtualization of the trivial link with two components. Hence, the usual Khovanov homology of L is $Kh(L) = (\mathbb{Z}_2)^4$ with gradings of generators equal to $(0, -2)$, $(0, 0)$, $(0, 0)$, $(0, 2)$.

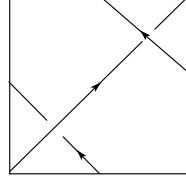


Figure 12: Link in the torus

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